math.NT/0103058

(submitted)

A lower bound in an approximation problem involving the zeros of the Riemann zeta function

Jean-François Burnol

March 2001

Abstract: We slightly improve the lower bound of Báez-Duarte, Balazard, Landreau and Saias in the Nyman-Beurling formulation of the Riemann Hypothesis as an approximation problem. We construct Hilbert space vectors which could prove useful in the context of the so-called "Hilbert-Pólya idea".

Author's affiliation:
Jean-François Burnol
Université de Nice - Sophia Antipolis
Laboratoire J.-A. Dieudonné
Parc Valrose
F-06108 Nice Cedex 02
France
electronic mail: burnol@math.unice.fr

Contents

1 Introduction **2**

2 The prediction error for a singular MA(q) **4**

3 Invariant analysis and a construction of Báez-Duarte **6**

4 The vectors $Y^{\lambda}_{s,k}$ **8**

5 The vectors $X^{\lambda}_{\rho,k}$ and completion of the proof **14**

6 Acknowledgements **16**

1 Introduction

In [8] and the subsequent paper [9], Connes gave a rather intrinsic construction of a Hilbert space intimately associated with the zeros of the Riemann zeta function on the critical line. But the zeros having multiplicities higher than a certain level (which is a parameter in Connes's construction), have (if they at all exist) their contributions limited to that level, and not to the extent given by their natural multiplicities. Thus subsists the problem of a natural definition of a so-called "Hilbert-Pólya space", with orthonormal basis indexed by the zeros ρ of ζ and integers k varying from 0 to $m_\rho - 1$ where m_ρ is the multiplicity of ρ. We do not solve that problem here but we do propose a rather natural construction of Hilbert space vectors $X^{\lambda}_{\rho,k}$, $\zeta(\rho) = 0$, $k < m_\rho$, which in the limit when the parameter λ goes to 0 become perpendicular

(when they correspond to distinct zeros. The vectors corresponding to a multiple root ρ are independent but need to be orthogonalized.) As in Connes's constructions these vectors live in a quotient space. Controlling the limit $\lambda \to 0$ to obtain a so-called Hilbert-Pólya space probably involves considerations from mathematical scattering theory (we have previously studied in [5], [6] some connections with the problems of $L-$functions.)

The context in which our construction takes place is that of the Nyman-Beurling formulation of the Riemann Hypothesis as an approximation problem [14], [3]. Let $K = L^2(]0, \infty[, dt)$ (over the complex numbers), let χ be the indicator function of the interval $]0, 1]$, and let ρ be the function "fractional part" (the letter ρ is also used to refer to a zero of the Riemann zeta function, hopefully no confusion will arise.) Let $0 < \lambda < 1$ and let $\mathcal{B}_\lambda$ be the sub-vector space of K consisting of the finite linear combinations of the functions $t \mapsto \rho(\frac{\theta}{t})$, for $\lambda \leq \theta \leq 1$.

Theorem 1.1 (Nyman [14], Beurling [3]) *The Riemann Hypothesis holds if and only if*

$$\chi \in \overline{\bigcup_{0<\lambda<1} \mathcal{B}_\lambda}$$

Actually we are following [2] here in using a slight variant of the original Nyman-Beurling formulation. It is a disappointing fact that this theorem can be proven without leading to any new information whatsoever on the zeros lying on the critical line (basically what is at works is the factorization of functions belonging to the Hardy space of a half-plane [12].) The following is thus rather remarkable:

Theorem 1.2 (Báez-Duarte, Balazard, Landreau and Saias [2]) *Let us write $D(\lambda)$ for the Hilbert-space distance* $\inf_{f \in \mathcal{B}_\lambda} \|\chi - f\|$. *We have*

$$\liminf_{\lambda \to 0} D(\lambda) \sqrt{\log(\frac{1}{\lambda})} \geq \sqrt{\sum_\rho \frac{1}{|\rho|^2}}$$

If the Riemann Hypothesis fails this result is true but trivial as the left-hand side then takes the value $+\infty$. So we will assume that the Riemann Hypothesis holds. The sum on the right-hand side is over all non-trivial zeros ρ of the zeta function, counted *only once* independently of their multiplicities m_ρ. We prove the following:

Theorem 1.3 *We have:*

$$\liminf_{\lambda\to 0} D(\lambda)\sqrt{\log(\frac{1}{\lambda})} \geq \sqrt{\sum_\rho \frac{m_\rho^2}{|\rho|^2}}$$

So the zeros are counted according to the *square* of their multiplicities. To prove this lower bound we will construct remarkable Hilbert space vectors $X^\lambda_{\rho,k}$, $\zeta(\rho) = 0$, $k < m_\rho$ and use them to control $D(\lambda)$. The following "toy-model" gives us reasons to expect that the lower bound in fact gives the exact order of decrease of $D(\lambda)$:

Theorem 1.4 *Let $Q(z) = \prod_\alpha (1 - \overline{\alpha}\cdot z)^{m_\alpha}$ be a polynomial of degree $q \geq 1$ will all its roots α on the unit circle (the root α having multiplicity m_α). Let $P(z)$ be an arbitrary polynomial. Let*

$$E(N,P) := \inf_{\deg(A)\leq N} \int_{S^1} |P(z) - Q(z)A(z)|^2 \frac{d\theta}{2\pi}$$

We have as N goes to infinity:

$$\lim\ N\,E(N,P) = \sum_\alpha m_\alpha^2\ |P(\alpha)|^2$$

2 The prediction error for a singular MA(q)

As motivation for our result we first consider a simpler approximation problem, in the context of the Hardy space of the unit disc rather than the Hardy space of a half-plane. Let $Q(z) = \prod_\alpha (1 - \overline{\alpha}\cdot z)^{m_\alpha}$ be a polynomial of degree $q \geq 1$ will all its roots α on the unit circle (the root

α having multiplicity m_α so that $q = \sum_\alpha m_\alpha$.) Let us define:

$$E(N) := \inf_{\deg(A)\leq N} \int_{S^1} |1 - Q(z)A(z)|^2 \frac{d\theta}{2\pi}$$

The measure $\frac{d\theta}{2\pi}$ is the rotation invariant probability measure on the circle S^1, with $z = \exp(i\theta)$. The minimum is taken over all complex polynomials $A(z)$ with degree at most N. We are guaranteed that $\lim_{N\to\infty} E(N) = 0$ as $Q(z)$ is an outer factor ([12]). More precisely:

Theorem 2.1 *As N goes to infinity we have:*

$$\lim \; N\, E(N) = \sum_\alpha m_\alpha^2$$

Note 2.2 In case $Q(z)$ has a root in the open unit disc then $E(N)$ is bounded below by a positive constant. In case $Q(z)$ has all its roots outside the open unit disc, then the result above holds but only the roots on the unit circle contribute. Finally if all its roots are outside the closed unit disc then the decrease is exponential: $E(N) = O(c^N)$, with $c < 1$.

The theorem, although not stated explicitely there, is easily extracted from the work of Grenander and Rosenblatt [11]. They state an $O(\frac{1}{N})$ result, in a much wider set-up than the one considered here (which is limited to simple-minded q-th order moving averages.) Unfortunately the $O(\frac{1}{N})$ bound is now believed not to be systematically true under their hypotheses (as is explained in [13]; I thank Professor W. Van Assche for pointing out this fact to me.) Nevertheless their technique of proof goes through smoothly in the case at hand and yields the exact asymptotic result as stated above. We only sketch briefly the idea, as nothing beyond the tools used in [11] is needed.

We point out in passing that it is of course possible to express $E(N)$ explicitely in terms of the Toeplitz determinants for the measure $d\mu = |Q(\exp(i\theta))|^2 \frac{d\theta}{2\pi}$. But already for an $MA(2)$ this gives rise to unwieldy computations.... Rather: let $\mathcal{P}_N$ be the vector space of polynomials of

degrees at most $N+q$, let $\mathcal{V}_N$ be the subspace of polynomials divisible by $Q(z)$, and let $\mathcal{W}_N$ be its q-dimensional orthogonal complement. Then $E(N)$ is the squared norm of the orthogonal projection of the constant function 1 to $\mathcal{W}_N$. A spanning set in $\mathcal{W}_N$ is readily identified: to each root α one associates $Y^N_{\alpha,0}$, $Y^N_{\alpha,1}$, ..., $Y^N_{\alpha,m_\alpha-1}$ defined as

$$Y^N_{\alpha,0} := 1 + \overline{\alpha}z + \ldots + \overline{\alpha}^{N+q} z^{N+q}$$

$$Y^N_{\alpha,1} := z + 2\overline{\alpha}z^2 + \ldots + (N+q)\overline{\alpha}^{N+q-1} z^{N+q}$$

and similarly for $k = 2, \ldots, m_\alpha - 1$. We can then express $E(N)$ using a Gram formula in terms of (the inverse) of the positive matrix (of fixed size $q \times q$ but depending on N) built with the scalar products of the Y's. It turns out that in the limit when N goes to infinity and after the rescaling $Y^N_{\alpha,k} \mapsto X^N_{\alpha,k} := N^{-k-1/2} Y^N_{\alpha,k}$ the Gram matrix decomposes into Cauchy blocks $(1/(i+j+1))_{0 \leq i,j < m_\alpha}$ of size m_α, one for each root α. It is known from Cauchy that the top-left element of the inverse matrix is m_α^2. This is how $\frac{\sum_\alpha m_\alpha^2}{N}$ arises, after keeping track of the scalar products $(1, X^N_{\alpha,k})$. Instead of the constant polynomial 1 we could have looked at the approximation rate to an arbitrary polynomial $P(z)$. The proof just sketched applies identically and gives the Theorem 1.4 from the Introduction.

3 Invariant analysis and a construction of Báez-Duarte

The Mellin transform $f(t) \mapsto \widehat{f}(s) = \int_{t>0} f(t) t^{s-1} dt$ establishes the Plancherel isometry between $K = L^2(]0, \infty[, dt)$ and $L^2(s = \frac{1}{2} + i\tau, \frac{d\tau}{2\pi})$, with inverse $F(s) \mapsto \int_{s=1/2+i\tau} F(s) t^{-s} \frac{d\tau}{2\pi}$. Let $a(s)$ be a measurable function of s (as a rule when using the letter s we implicitely assume $\mathrm{Re}(s) = \frac{1}{2}$. We will use letters w and z for general complex numbers.) If $a(s)$ is essentially bounded then $F(s) \mapsto a(s)F(s)$ defines a bounded operator on K which commutes with the unitary group $D_\theta : f(t) \mapsto \frac{1}{\sqrt{\theta}} f(\frac{t}{\theta})$, and all bounded operators commuting with the D_θ $(0 < \theta < \infty)$ are obtained in such a manner. More generally all *closed* invariant operators are associated to a measurable multiplier $a(s)$ (finite almost everywhere, but not necessarily essentially bounded). For the details of this technical statement, see [7].

For example the Hardy averaging operator $M: f(t) \mapsto \frac{1}{t}\int_{]0,t]} f(u)du$ corresponds to the spectral multiplier $\frac{1}{1-s}$. The operator $1-M$ corresponds to the spectral multiplier $\frac{s}{s-1}$ and is thus unitary. Another (see [4]) remarkable invariant operator is the (even) "Gamma" operator $\Gamma_+ = \mathcal{F}_+ I$. Here I is the inversion $f(t) \mapsto \frac{1}{t}f(\frac{1}{t})$ and $\mathcal{F}_+$ is the additive Fourier transform as applied to even functions (the cosine transform). The multiplier associated to Γ_+ is the (Tate) function

$$\gamma_+(s) = \pi^{\frac{1}{2}-s}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \frac{\zeta(1-s)}{\zeta(s)} = 2^{1-s}\pi^{-s}\cos(\frac{\pi\, s}{2})\Gamma(s) = (1-s)\int_0^\infty u^{s-1}\,\frac{\sin(2\pi u)}{\pi u}\,du$$

A further invariant operator is the operator U introduced by Báez-Duarte [1] in connection with the Nyman-Beurling formulation of the Riemann Hypothesis: its spectral multiplier is $\frac{s}{1-s}\frac{\zeta(1-s)}{\zeta(s)}$, so $U = (M-1)\mathcal{F}_+ I = \mathcal{F}_+ I(M-1)$.

From the results recalled above on invariant operators, we see that invariant orthogonal projectors correspond to indicator functions of measurable sets on the critical line. So a function $f(t)$ is such that its multiplicative translates $D_\theta(f)$ $(0 < \theta < \infty)$ span K if and only if $F(s) = \widehat{f}(s)$ is almost everywhere non-vanishing (Wiener's L^2-Tauberian Theorem.) In that case the phase function

$$U_f(s) = \frac{\overline{F(s)}}{F(s)}$$

is almost everywhere defined and of modulus 1. It thus corresponds to an invariant unitary operator, also denoted U_f.

Let us introduce the *anti-unitary* "time-reversal" operator J acting on K as $g \mapsto \overline{I(g)}$. The operator U_f commutes with the contractions-dilations, is unitary, and sends f to $J(f)$. We call this the *Báez-Duarte construction* as it appears in [1] (up to some non-essential differences) in relation with the Nyman-Beurling problem (the phase function arises in other contexts, especially in scattering theory.)

To relate this with the operator $U = (M-1)\mathcal{F}_+ I$, one needs the formula

$$\frac{\zeta(s)}{s} = -\int_0^\infty \rho(\frac{1}{t})t^{s-1}dt$$

which is fundamental in the Nyman-Beurling context. This formula shows that U is the phase operator associated with $\rho(\frac{1}{t})$.

Generally speaking, the operators U_f are related to the Hardy spaces $\mathbb{H}^2 = L^2(]0,1], dt)$ and $\mathbb{H}^{2\perp} = L^2([1,\infty[, dt)$ (we will also use the notation $\mathbb{H}^2$ for the Mellin transform of $L^2(]0,1], dt)$.) Indeed the time-reversal J is an isometry (anti-unitary) between $\mathbb{H}^2$ and $\mathbb{H}^{2\perp}$. Let us assume that the function f belongs to $\mathbb{H}^2$. The operator U_f has the same effect as J on f, but contrarily to J is an *invariant* operator. This puts the space $\mathcal{B}_\lambda(f)$ (of finite linear combinations of contractions $D_\theta(f)$ for $\lambda \leq \theta \leq 1$) isometrically in a new light as a subspace of $L^2([\lambda,\infty[, dt)$. The marvelous thing is that in this new incarnation it appears to be sometimes possible to find vectors orthogonal to $\mathcal{B}_\lambda(f)$ and thus to get some control on $\mathcal{B}_\lambda(f)$ as λ decreases (as in the Grenander-Rosenblatt method.)

4 The vectors $Y_{s,k}^\lambda$

To get started on this we first replace the L^2 function $-\frac{\zeta(s)}{s}$ with an element of $\mathbb{H}^2$. This is elementary:

Proposition 4.1 ([6], [10]) *The function $Z(s) = \frac{s-1}{s}\frac{\zeta(s)}{s}$ belongs to $\mathbb{H}^2$. Its inverse Mellin transform $A(t)$ is given by the formula*

$$A(t) = [\frac{1}{t}]\log(t) + \log([\frac{1}{t}]\,!) + [\frac{1}{t}]$$

One has (for $0 < t \leq 1$) $A(t) = \frac{1}{2}\log(\frac{1}{t}) + O(1)$.

The Báez-Duarte construction will then associate to $A(t)$ the operator V with spectral multiplier

$$V(s) = \left(\frac{s}{1-s}\right)^3 \frac{\zeta(1-s)}{\zeta(s)}$$

so that

$$V = (1 - M)^2 \cdot U$$

This last representation will prove useful as it allows to use the formulae related to U from [1] and [2]. Let $\mathcal{C}_\lambda$ $(0 < \lambda < 1)$ be the sub-vector space of $\mathbb{H}^2$ of linear combinations of the contractions $D_\theta(A)$ for $\lambda \leq \theta \leq 1$. The function $\frac{s-1}{s}\frac{1}{s} = \frac{1}{s} - \frac{1}{s^2}$ is the Mellin transform of $\chi_1(t) := (1 + \log(t))\chi(t)$. The quantity $D(\lambda)$ considered by Báez-Duarte, Balazard, Landreau and Saias is thus the Hilbert space distance between $\chi_1(t)$ and $\mathcal{C}_\lambda$. To bound it from below we will exhibit remarkable Hilbert space vectors $X^\lambda_{\rho,k}$ indexed by the zeros of the Riemann zeta function and perpendicular to $\mathcal{C}_\lambda$. We then compute the exact asymptotics of the orthogonal projection of χ_1 to the vector spaces spanned by the $X^\lambda_{\rho,k}$, for a finite set of roots, exactly as in the Grenander-Rosenblatt method.

To each complex number w and natural integer $k \geq 0$ we associate the funtion $\psi_{w,k}(t) = (\log(\frac{1}{t}))^k \, t^{-w} \, \chi(t)$ on $]0, \infty[$. For $\mathrm{Re}(w) < 1$ it is integrable, for $\mathrm{Re}(w) < \frac{1}{2}$ it is in K. Let Q_λ be the orthogonal projector from K onto $L^2([\lambda, \infty[)$. The main point of this paper is the following:

Theorem and Definition 4.2 *For each $0 < \lambda \leq 1$, each s on the critical line, and each integer $k \geq 0$ the L^2-limit in K of $V^{-1}Q_\lambda V(\psi_{w,k})$ exists as w tends to s from the left half-plane:*

$$Y^\lambda_{s,k} := \underset{\substack{w \to s \\ \mathrm{Re}(w) < \frac{1}{2}}}{\mathrm{l.i.m.}} \; V^{-1} Q_\lambda V(\psi_{w,k})$$

For each $\lambda \leq \theta \leq 1$ the scalar products between $D_\theta(A)$ and the vectors $Y^\lambda_{s,k}$ are:

$$\lambda \leq \theta \leq 1 \;\Rightarrow\; (D_\theta(A), Y^\lambda_{s,k}) = \left(-\frac{d}{ds}\right)^k \; \theta^{s-\frac{1}{2}} Z(s)$$

Note 4.3 The proof shows the existence of an analytic continuation in w accross the critical line, but we shall not make use of this fact.

Clearly one has the following statement as an immediate consequence:

Corollary 4.4 *Let $0 < \lambda < 1$. The vector $Y^{\lambda}_{s,k}$ is perpendicular to $\mathcal{C}_\lambda$ if and only if $\zeta^{(j)}(s) = 0$ for all $j \le k$, if and only if s is a zero ρ of the zeta function and $k < m_\rho$.*

Note 4.5 Our scalar products (f, g) are complex linear in the first factor and conjugate-linear in the second factor.

Note 4.6 The operator $\frac{d}{ds}$ when applied to a not necessarily analytic function on the critical line is defined to act as $\frac{1}{i}\frac{d}{d\tau}$ (where $s = \frac{1}{2} + i\tau$.)

Proof The proof of existence will be given later. Here we check the statement involving the scalar product, assuming existence. The following holds for $\lambda \le \theta \le 1$ and $\mathrm{Re}(w) < \frac{1}{2}$:

$$\begin{aligned}
(V^{-1}Q_\lambda V(\psi_{w,k}), D_\theta(A)) &= (Q_\lambda V(\psi_{w,k}), V D_\theta(A)) \\
&= (Q_\lambda V(\psi_{w,k}), D_\theta \cdot V(A)) \\
&= (V(\psi_{w,k}), Q_\lambda \cdot D_\theta \cdot J(A)) \\
&= (V(\psi_{w,k}), D_\theta \cdot V(A)) \\
&= (\psi_{w,k}\ , D_\theta(A)) \\
&= \left(\frac{d}{dw}\right)^k (\psi_{w,0}\ , D_\theta(A)) \\
&= \left(\frac{d}{dw}\right)^k (D_\theta^{-1}(t^{-w}\,\chi(t)), A) \\
&= \left(\frac{d}{dw}\right)^k (\theta^{1/2-w} t^{-w}\,\chi(\theta t), A) \\
&= \left(\frac{d}{dw}\right)^k \theta^{1/2-w} \int_{]0,1]} t^{-w}\overline{A(t)}\,dt
\end{aligned}$$

Taking the limit when $w \to s$ gives

$$\begin{aligned}
(Y^{\lambda}_{s,k}, D_\theta(A)) &= \left(\frac{d}{ds}\right)^k \theta^{1/2-s} \int_{]0,1]} t^{-s}\overline{A(t)}\,dt \\
&= \left(\frac{1}{i}\frac{d}{d\tau}\right)^k \theta^{-i\tau} \int_{]0,1]} t^{-\frac{1}{2}-i\tau}\,\overline{A(t)}\,dt
\end{aligned}$$

Taking the complex conjugate:

$$
\begin{aligned}
(D_\theta(A), Y_{s,k}^\lambda) &= \left(i\frac{d}{d\tau}\right)^k \theta^{i\tau} \int_{]0,1]} t^{-\frac{1}{2}+i\tau}\, A(t)\, dt \\
&= \left(-\frac{d}{ds}\right)^k \theta^{s-\frac{1}{2}} \int_{]0,1]} t^{s-1}\, A(t)\, dt \\
&= \left(-\frac{d}{ds}\right)^k \theta^{s-\frac{1}{2}} Z(s)
\end{aligned}
$$

which completes the proof (assuming existence.) •

To prove the existence we will use in an essential manner the key **Lemme 6** from [2]. We have seen that $V = (1-M)^2 U$ where M is the Hardy averaging operator and U the Báez-Duarte operator. The spectral function $U(s)$ extends to an analytic function $U(w)$ in the strip $0 < \mathrm{Re}(w) < 1$. We need pointwise expressions for $V(\psi_{w,k})(t)$, $t > 0$ (at first only $\mathrm{Re}(w) < \frac{1}{2}$ is allowed here). Thanks to the general study of U given in [1], we know that for $\mathrm{Re}(w) < \frac{1}{2}$ the vector $U(\psi_{w,k})$ in K is given as the following limit in square mean:

$$
\underset{\delta \to 0}{\mathrm{l.i.m.}} \int_\delta^1 (\log(\frac{1}{v}))^k\, v^{-w}\, \frac{d}{dv} \frac{\sin(2\pi t/v)}{\pi t/v}\, dv
$$

Following [2], with a slight change of notation, we now study for each complex number w with $\mathrm{Re}(w) < 1$ (and each integer $k \geq 0$) the *pointwise* limit as a function of $t > 0$ for $\delta \to 0$:

$$
\varphi_{w,k}(t) := \lim_{\delta \to 0} \int_\delta^1 (\log(\frac{1}{v}))^k\, v^{-w}\, \frac{d}{dv} \frac{\sin(2\pi t/v)}{\pi t/v}\, dv
$$

Theorem 4.7 ([2]) *Let $k = 0$. For each $t > 0$ and $\mathrm{Re}(w) < 1$ the pointwise limit defining $\varphi_{w,0}(t)$ exists. It is holomorphic in w for each fixed t. When w is restricted to a compact set in $\mathrm{Re}(w) < 1$, one has uniformly in w the bound $\varphi_{w,0}(t) = O(\frac{1}{t})$ on $[1, \infty[$. Uniformly with respect to w satisfying $0 < \mathrm{Re}(w) < 1$ one has $\varphi_{w,0}(t) = U(w)\, t^{-w} + O(1)$ on $0 < t \leq 1$.*

Proof Everything is either stated explicitely in [2], Lemme 6 and Lemme 4, or follows from their proofs. We will give more details for $k \geq 1$ as this is not treated in [2]. •

Corollary 4.8 *For each w in the critical strip $0 < \mathrm{Re}(w) < 1$ the Hardy operator $M : f(t) \to \frac{1}{t}\int_0^t f(v)\,dv$ can be applied arbitrarily many times to $\varphi_{w,0}(t)$. The functions $M^L(\varphi_{w,0})$ ($L \in \mathbb{N}$) are $O(\frac{(1+\log(t))^L}{t})$ on $[1,\infty[$, uniformly with respect to w when it is restricted to a compact subset of the open strip, and satisfy on $t \in]0,1]$ the estimate $M^L(\varphi_{w,0})(t) = \left(\frac{1}{1-w}\right)^L U(w)\,t^{-w} + O(1)$, uniformly with respect to w.*

Proof A simple recurrence. •

We thus obtain:

Corollary 4.9 *The vectors $Y^\lambda_{s,0}$ exist (for $\mathrm{Re}(s) = \frac{1}{2}$). One has the estimates:*

$$V(Y^\lambda_{s,0})(t) = O(\frac{(1+\log(t))^2}{t}) \qquad (t \in [1,\infty[)$$

$$V(Y^\lambda_{s,0})(t) = V(s)\,t^{-s} + O(1) \qquad (\lambda < t \leq 1)$$

$$V(Y^\lambda_{s,0})(t) = 0 \qquad (0 < t < \lambda)$$

uniformly with respect to s when its imaginary part is bounded.

Theorem 4.10 *Let $k \geq 1$. For each $t > 0$ and $\mathrm{Re}(w) < 1$ the pointwise limit defining $\varphi_{w,k}(t)$ exists. It is holomorphic in w for each fixed t. When w is restricted to a compact set in $\mathrm{Re}(w) < 1$, one has uniformly in w the bound $\varphi_{w,k}(t) = O(\frac{1}{t})$ on $[1,\infty[$. Uniformly for $0 < \mathrm{Re}(w) < 1$ one has $\varphi_{w,k}(t) = \left(\frac{d}{dw}\right)^k (U(w)\,t^{-w}) + O(1)$ on $0 < t \leq 1$.*

Proof The formula defining $\varphi_{w,k}(t)$ is equivalent to (after integration by parts and the change of variable $u = 1/v$):

$$\varphi_{w,k}(t) = \lim_{\Lambda\to\infty} \frac{1}{\pi\,t}\int_1^\Lambda (k + w\log(u))\,(\log(u))^{k-1}\;u^{w-1}\sin(2\pi t\,u)\,\frac{du}{u}$$

This proves the existence of $\varphi_{w,k}(t)$, its analytic character in w, and the uniform $O(\frac{1}{t})$ bound on $[1,\infty[$. The formula can be rewritten as:

$$\varphi_{w,k}(t) = \left(\frac{d}{dw}\right)^k \frac{w}{\pi\, t} \int_1^\infty u^{w-1} \sin(2\pi t\, u)\, \frac{du}{u}$$

When w is in the critical strip the integral $\int_0^\infty u^{w-1} \sin(2\pi t\, u)\, \frac{du}{u}$ is absolutely convergent and its value is $t^{1-w} \int_0^\infty u^{w-1} \sin(2\pi u)\, \frac{du}{u} = \frac{1}{1-w}(2\pi\, t)^{1-w} \cos(\frac{\pi w}{2})\Gamma(w)$ from well-known integral formulae, so that:

$$\varphi_{w,k}(t) = \left(\frac{d}{dw}\right)^k \left(\frac{w}{1-w} 2^{1-w}\pi^{-w} \cos(\frac{\pi w}{2})\Gamma(w) t^{-w} - \frac{w}{\pi\, t} \int_0^1 u^{w-1} \sin(2\pi t\, u)\, \frac{du}{u}\right)$$

The first term is $\left(\frac{d}{dw}\right)^k (U(w)\, t^{-w})$ and the second term can be explicitely evaluated using the series expansion of $\sin(2\pi t\, u)$ with the final result

$$\varphi_{w,k}(t) = \left(\frac{d}{dw}\right)^k (U(w)\, t^{-w}) + 2(-1)^k\, k! \sum_{j\geq 1} (-1)^j \frac{(2\pi t)^{2j}}{(2j+1)!} \frac{2j}{(w+2j)^{k+1}}$$

which shows $\varphi_{w,k}(t) = \left(\frac{d}{dw}\right)^k (U(w)\, t^{-w}) + O(1)$, on $0 < t \leq 1$, uniformly for $0 < \mathrm{Re}(w) < 1$. •

As was the case for $k = 0$ we then deduce that the Hardy operator can be applied arbitrarily many times to $\varphi_{w,k}$ for $0 < \mathrm{Re}(w) < 1$. The existence of the $Y^\lambda_{s,k}$ follows.

Theorem 4.11 *Let $k \geq 0$. The vectors $Y^\lambda_{s,k}$ exist (for $\mathrm{Re}(s) = \frac{1}{2}$). One has the estimates:*

$$\begin{aligned} V(Y^\lambda_{s,k})(t) &= O(\frac{(1+\log(t))^2}{t}) \qquad (t \in [1,\infty[) \\ V(Y^\lambda_{s,k})(t) &= \left(\frac{d}{ds}\right)^k (V(s)\, t^{-s}) + O(1) \qquad (\lambda < t \leq 1) \\ V(Y^\lambda_{s,k})(t) &= 0 \qquad (0 < t < \lambda) \end{aligned}$$

the implied constants are independent of λ and are uniform with respect to s when its imaginary part is bounded.

Proof Clearly a corollary to 4.10. •

5 The vectors $X^\lambda_{\rho,k}$ and completion of the proof

Definition 5.1 *Let $0 < \lambda < 1$. To each zero ρ of the Riemann zeta function on the critical line, of multiplicity m_ρ, and each integer $0 \le k < m_\rho$ we associate the Hilbert space vector*

$$X^\lambda_{\rho,k} := \left(\log(\frac{1}{\lambda})\right)^{-\frac{1}{2}-k} \cdot Y^\lambda_{\rho,k}$$

where $Y^\lambda_{\rho,k} = \text{l.i.m.}_{w\to s} V^{-1} Q_\lambda V(\psi_{w,k})$, V is the unitary operator $(M-1)^3 \mathcal{F}_+ I$, Q_λ is orthogonal projection to $L^2([\lambda,\infty[, dt)$, and $\psi_{w,k}(t) = (\log(\frac{1}{t}))^k\, t^{-w}\, \chi(t)$.

Note 5.2 Of course there is no reason except psychological to allow only zeros of the Riemann zeta function at this stage.

Theorem 5.3 *As λ decreases to 0 one has:*

$$\begin{aligned} \lim_{\lambda\to 0} (X^\lambda_{\rho_1,k}, X^\lambda_{\rho_2,l}) &= 0 \qquad (\rho_1 \neq \rho_2) \\ \lim_{\lambda\to 0} (X^\lambda_{\rho,k}, X^\lambda_{\rho,l}) &= \frac{1}{k+l+1} \end{aligned}$$

Proof To establish this we first consider, for $\mathrm{Re}(s_1) = \mathrm{Re}(s_2) = \frac{1}{2}$:

$$\int_\lambda^1 \left(\log(\frac{1}{t})\right)^{j_1} t^{-s_1} \left(\log(\frac{1}{t})\right)^{j_2} t^{-(1-s_2)}\, dt$$

If $s_1 \neq s_2$ an integration by parts shows that it is $O\left(\log(\frac{1}{\lambda})\right)^{j_1+j_2}$. On the other hand when $s_1 = s_2$ its exact value is $\frac{1}{j_1+j_2+1}\left(\log(\frac{1}{\lambda})\right)^{j_1+j_2+1}$. With this information the theorem follows directly from 4.11 as (for example) the leading divergent contribution as $\lambda \to 0$ to $\left(V(Y^\lambda_{s,k}),\ V(Y^\lambda_{s,l})\right)$ is $V(s)\overline{V(s)} \int_\lambda^1 \left(\frac{d}{ds}\right)^k t^{-s}\ \overline{\left(\frac{d}{ds}\right)^l t^{-s}}\, dt$ which gives $\frac{1}{k+l+1}\left(\log(\frac{1}{\lambda})\right)^{k+l+1}$. The rescaling $Y \mapsto X$ is chosen so that a finite limit for $(X^\lambda_{\rho,k}, X^\lambda_{\rho,l})$ is obtained. As the scalar products involving distinct zeros have a smaller divergency, the rescaling let them converge to 0. •

Theorem 5.4 *Let $\chi_1(t) = (1+\log(t))\chi(t)$. As λ decreases to 0 one has:*

$$\begin{aligned}
\lim_{\lambda\to 0} \sqrt{\log(\frac{1}{\lambda})}\,(\chi_1, X^\lambda_{\rho,k}) &= 0 \qquad (k \geq 1) \\
\lim_{\lambda\to 0} \sqrt{\log(\frac{1}{\lambda})}\,(\chi_1, X^\lambda_{\rho,0}) &= \frac{\rho-1}{\rho^2}
\end{aligned}$$

Proof We have $(1-M)\chi_1 = \chi$, and $V = (1-M)^2\,U$ so $V\chi_1 = (1-M)\,U\chi$. From [1] we know that $U\chi$ is $\frac{\sin(2\pi t)}{\pi t}$ so $V\chi_1$ is the function $\frac{\sin(2\pi t)}{\pi t} - \frac{1}{t}\int_0^t \frac{\sin(2\pi v)}{\pi v}\,dv$. It is thus $0(t^2)$ as $t \to 0$, and from 4.11 we then deduce that the scalar products $(\chi_1, Y^\lambda_{\rho,k})$ admit finite limits as $\lambda \to 0$. This settles the case $k \geq 1$. For $k = 0$, one uses the uniformity with respect to w in 4.7 to get

$$\lim_{\lambda\to 0}(\chi_1, Y^\lambda_{\rho,0}) = \lim_{w\to\rho}(\chi_1, \varphi_{w,0})$$

which gives $\lim_{w\to\rho}\int_0^1 (1+\log(t))\,t^{w-1}\,dt = \frac{1}{\rho} - \frac{1}{\rho^2} = \frac{\rho-1}{\rho^2}$. •

We can now conclude the proof of our estimate.

Theorem 5.5 *We have:*

$$\liminf_{\lambda\to 0} D(\lambda)\sqrt{\log(\frac{1}{\lambda})} \geq \sqrt{\sum_\rho \frac{m_\rho^2}{|\rho|^2}}$$

Proof Let R be a non-empty finite set of zeros. We showed that $D(\lambda)$ is the Hilbert space distance from χ_1 to $\mathcal{C}_\lambda$, and that the vectors $X^\lambda_{\rho,k}$ for $0 \leq k < m_\rho$ are perpendicular to $\mathcal{C}_\lambda$. So $D(\lambda)$ is bounded below by the norm of the orthogonal projection of χ_1 to the finite-dimensional vector space H_R spanned by the vectors $X^\lambda_{\rho,k}$, $0 \leq k < m_\rho$, $\rho \in R$. This is given by a well-known formula involving the inverse of the Gram matrix of the $X^\lambda_{\rho,k}$'s as well as the scalar products $(\chi_1, X^\lambda_{\rho,k})$. From 5.3 the Gram matrix converges to diagonal blocks, one for each zero, given by Cauchy matrices of sizes $m_\rho \times m_\rho$. From Cauchy we know that the top-left element of the inverse matrix is m_ρ^2. Combining this with the scalar products evaluated in 5.4 we get that the

squared norm of the orthogonal projection of χ_1 to H_R is asymptotically equivalent as $\lambda \to 0$ to $\frac{\sum_{\rho \in R} \frac{m_\rho^2}{|\rho|^2}}{\log(\frac{1}{\lambda})}$. The proof is complete. •

We can apply our strategy to a fully singular MA(q) on the unit circle. The relevant Báez-Duarte phase operator will then be (up to a non-important constant of modulus 1) the operator of multiplication by z^{-q} and it is apparent that this leads to a proof equivalent to the one we gave in our previous discussion, inspired by [11]. In the case of the Nyman-Beurling approximation problem for the zeta funtion, we expect in the quotient of $\mathbb{H}^2$ by $\overline{\mathcal{C}_\lambda}$ a "continuous spectrum" additionally to the "discrete spectrum" provided by the (projection to $\mathbb{H}^2$ of the) $X_{\rho,k}^\lambda$'s, $\zeta(\rho) = 0$, $k < m_\rho$. It is tempting to speculate that the rescaling will kill this continuous part as $\lambda \to 0$, so that in the end only subsists a so-called "Hilbert-Pólya" space. This would appear to require 5.5 to give the exact order of decrease of the quantity $D(\lambda)$ and the numerical explorations reported by Báez-Duarte, Balazard, Landreau and Saias in [2] seem to support this.

6 Acknowledgements

I had been looking for the vectors $X_{\rho,k}^\lambda$, and with the **Lemme 6** of [2] they were suddenly there. I thank Michel Balazard for giving me copies of [1] and [2] in preprint form.

References

[1] L. Báez-Duarte, *A class of invariant unitary operators*, Adv. in Math. **144** (1999), 1-12.

[2] L. Báez-Duarte, M. Balazard, B. Landreau and E. Saias, *Notes sur la fonction ζ de Riemann 3*, Adv. in Math. **149** (2000), 130-144.

[3] A. Beurling, *A closure problem related to the Riemann Zeta–function*, Proc. Nat. Acad. Sci. **41** (1955), 312-314.

[4] J.-F. Burnol, *"Sur les formules explicites I: analyse invariante"*, C. R. Acad. Sci. Paris **331** (2000), Série I, 423-428.

[5] J.-F. Burnol, *Scattering on the p-adic field and a trace formula*, International Mathematical Research Notices 2000:**2** (2000), 57-70.

[6] J.-F. Burnol, *An adelic causality problem related to abelian $L-$functions*, Journal of Number Theory. To appear.

[7] J.-F. Burnol, *Quaternionic gamma functions and their logarithmic derivatives as spectral functions*, Mathematical Research Letters. To appear.

[8] A. Connes, *Formule de trace en géométrie non-commutative et hypothèse de Riemann*, C. R. Acad. Sci. Paris **323** (1996), Série I, 1231-1236.

[9] A. Connes, *Trace formula in non-commutative Geometry and the zeros of the Riemann zeta function*, Selecta Math. (N.S.) **5** (1999) , no. 1, 29–106.

[10] W. Ehm, *A family of probability densities related to the Riemann zeta function*, manuscript (2000), 12 pp.

[11] U. Grenander, M. Rosenblatt, *An extension of a theorem of G. Szegö and its application to the study of stochastic processes*, Trans. Amer. Math. Soc. **76** (1954), 112-126.

[12] K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Inc. (1962). (Dover Pub., 1988).

[13] P. Nevai, *Géza Freud, Orthogonal Polynomials and Christoffel Functions. A Case Study*, Journ. Approx. Theory **48** (1986), 3-167.

[14] B. Nyman, *On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces*. Thesis, University of Uppsala, 1950. 55 pp.